Models of Set Theory I – Summer 2017

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Problem 5 Prove the following statements for all class terms S, T, W.

(a) [1 point]: $(S = T)^W \Longleftrightarrow S^W = T^W$.

(b) [1 point]: $(S \in T)^W \Longleftrightarrow S^W \in T^W$.

Problem 6 [4 points]: Working in ZFC, let $W = V \setminus V_\omega$. Examine which axioms and schemes of ZFC hold in W.

Problem 7 We say that a sequence $(C_\alpha \mid \alpha \in \mathrm{Ord})$ is a *continuous hierarchy* if the following conditions hold.

(i) $V = \bigcup_{\alpha \in \mathrm{Ord}} C_\alpha$.

(ii) $C_\alpha \subseteq C_\beta$ for all ordinals $\alpha < \beta$.

(iii) $C_\beta = \bigcup_{\alpha < \beta} C_\alpha$ for all limit ordinals β.

Prove the following statements.

(a) [2 points]: If $(C_\alpha \mid \alpha \in \mathrm{Ord})$ and $(D_\alpha \mid \alpha \in \mathrm{Ord})$ are continuous hierarchies, then there is a club E in Ord such that $C_\alpha = D_\alpha$ for all $\alpha \in E$.

(b) [2 points]: $H_\kappa = V_\kappa$ for every inaccessible cardinal κ.

(c) [2 points]: If κ is an inaccessible cardinal, then there is an uncountable cardinal $\lambda < \kappa$ with $H_\lambda = V_\lambda$.

Problem 8 We define the following notions for any infinite cardinal κ and any partial order $\mathbb{P} = (P, \leq)$.

(i) p, q are *compatible in* $\mathbb{P}$ if there is some $r \leq p, q$.

(ii) An *antichain in* $\mathbb{P}$ is a set of pairwise incompatible elements of $\mathbb{P}$.

(iii) $\mathbb{P}$ has the *κ-chain condition* (κ-c.c.) if and only if there is no antichain in $\mathbb{P}$ of size κ.

If λ is an infinite cardinal, let $\mathbb{C}_\lambda$ denote the partial order $\mathbb{C}_\lambda = (C_\lambda, \supseteq)$, where

$$C_\lambda = \{p\colon \alpha \to 2 \mid \alpha < \lambda\}.$$

If $\mathbb{Q} = (Q, \leq)$ is any partial order, let $\mathbb{Q} \times \mathbb{Q}$ denote the product partial order with domain $Q \times Q$ that is defined by

$$(a, b) \leq (c, d) \Leftrightarrow a \leq c \wedge b \leq d.$$

(a) [4 points]: Determine the least infinite cardinal κ such that $\mathbb{C}_{\omega_1}$ has the κ-c.c.

(b) [4 points]: Show that $\mathbb{C}_\omega$ and $\mathbb{C}_\omega^2$ are isomorphic partial orders.